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ON THE GENERALIZED THERMOELASTIC SURFACE WAVES FOR MATERIALS CHARACTERIZATION

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ABSTRACT

In this article analysis for propagation of surface waves on a heat conducting transversely isotropic half space rigidly bonded to a thermoelastic transversely isotropic substrate half space of different material in the context of generalized thermoelasticity is presented. The total system is in contact with a fluid space at uniform temperature. Considering the wave that propagates along the fluid layer interface, analytical results are obtained in the closed form. Results for heat conducting thermoelastic solids possessing higher symmetry materials properties such as cubic and isotropic are the particular cases of the general solutions obtained. On setting either the thickness of the layer equal to zero or by setting the properties of the layer equal to corresponding properties of the substrate, results for half space can be deduced. Furthermore, in the absence of the fluid i.e. setting the density of the fluids equal to zero, results for dry medium can be obtained. The model developed will be of value in material characterization and others quantitative information on thermo-mechanical, strength related properties of advanced materials. Relevant results of previous investigations classical and coupled thermoelasticity are deduced as special cases.

KEYWORDS: Surface Waves, Thermoelasticity, Classical theory, Multilayered, Coupled, Thermal Relaxation Time.

INTRODUCTION

Classical theory of thermoelasticity, based on the Fourier law, implies an immediate response to a temperature gradient and leads to a parabolic differential equation for the evolution of the temperature, which is physically impractical phenomenon of instantaneous heat propagation. This is the well-known limitation of the classical theory of thermoelasticity, when applied to number of practically important problems such as dealing with sudden heat inputs or extremely low temperature regimes. In contrast, when relaxation effects are taken into account in the constitutive equation describing the heat flux, as, for instance, in the Maxwell-Cattaneo equation, one has a hyperbolic equation, which implies a finite speed for heat transport. The classical theory of dynamic thermoelasticity that takes into account the coupling effects between temperature and strain fields involves the infinite thermal wave speed. The theories of generalized thermoelasticity have been developed in an attempt to eliminate this paradox of infinite velocity of thermal propagation. Lord and Shulman [1] have formulated a generalized dynamical theory of thermoelasticity by using a form of the heat conduction equation that includes the time needed for acceleration of the heat flow. This new theory which is named as the 'Generalized Theory of thermoelasticity' eliminates the paradox of an infinite velocity of propagation and admit finite speed for the propagation of thermoelastic disturbances received much attention in recent years [2-5].

Recently, Banerjee and Pao [6] have extended this theory to anisotropic heat conducting elastic materials. Dhaliwal and Sherief [7] treated the problem in more systematic manner. They derived governing field equations of generalized thermoelastic media and proved that these equations are unique.

Propagation of generalized thermoelastic free waves in plate of anisotropic media has been studied by Verma [8], Verma and Hasebe [9,10], Chadwick and Seet [12], Verma and Hasebe [8] studied the wave propagation in plates of general anisotropic media in generalized thermoelasticity. In the present paper we study the propagation of harmonic thermoelastic surface waves on a heat conducting transversely isotropic half space rigidly bonded to a thermoelastic transversely isotropic substrate half space of different material. The total system is in contact with a fluid space at uniform temperature. Considering the wave that propagates along the fluid layer interface, analytical results are obtained in the closed form. Results for thermoelastic solids possessing cubic and isotropic materials are the special cases of the general solutions obtained. On setting either the thickness of the layer equal to zero or by setting the properties of the layer equal to corresponding properties of the substrate, results for half space can be deduced. Furthermore, in the absence of the fluid that is on setting the density of the fluid equal to zero, results for dry medium can be obtained. The model developed will be of value in material characterization and others quantitative information on thermomechanical, strength related properties of advanced materials. Relevant results of previous investigations and coupled thermoelasticity are deduced as special cases.

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FORMULATION

Consider a heat conducting transversely-isotropic plate having the thickness *d* rigidly attached to a thermoelastic transversely-isotropic solid half space of similar or different material and separating the latter from a fluid half space assumed at a uniform temperature. In order to study the characteristics of the thermoelastic surface wave propagating along the plate-fluid interface two-dimensional coordinate system x_i, $i=1,\ 2$, which has its origin at the substrate-plate interface such that x_i denotes the propagation direction and x_2 is normal to the interface is considered. The latter thus occupy the space $0 \le x_2 \le d$. All motions will be independent of x_3 -direction with this choice of coordinate system,

The governing equations of motion and heat conduction in the context of generalized theory of thermoelasticity in the absence of bodyforces and heat sources of theplate are [7]

$$\frac{\partial \sigma_{ij}}{\partial x_j} = \frac{\partial^2 u_i}{\partial t^2},\quad i,j=1,2, \tag{1}$$

$$\sigma_{ij} = C_{ijkl} e_{kl} - \beta_{ij} T, \tag{2}$$

$$\beta_{ij} = C_{ijkl}\alpha_{kl},\ i,j,k,l=1,\ 2 \tag{3}$$

$$K_{ij}\frac{\partial^2 T}{\partial x_i \partial x_j} - C_e\left(\frac{\partial T}{\partial t} + \tau_0 \frac{\partial^2 T}{\partial t^2}\right) = T_0 \beta_{ij}\left(\frac{\partial e_{kk}}{\partial t} + \tau_0 \frac{\partial^2 e_{kk}}{\partial t^2}\right), \tag{4}$$

where the symbols have been defined in the above cited reference. The total system is in contact with a fluid space. The analytical results are presented in the closed form. Results for heat conducting solids possessing cubic and isotropic material symmetries will be found as special cases of the general solutions by involving the appropriate restrictions on the properties. Furthermore, results for a half space can be deduced by either setting the thickness of the layer or by setting the properties of the layer equal to their corresponding properties of the substrate. In all cases, results for the dry medium can be obtained by taking the density of the fluid to zero.

On Specializing the equations (1)-(4) to transversely isotropic materials, with the appropriate interfacial continuity conditions for rigid boundary between the plate and substrate are

$$\sigma_{i2} = \sigma_{i2}^{(s)},\quad u_i = u_i^{(s)}\quad i = 1,2, \tag{5}$$

$$T = T^{(s)}\quad \text{at}\ x_2 = 0\,. \tag{6}$$

Here the subscript (s) designates the substrate whereas the layer is identified by omission of the subscript. Finally at the fluid plate interface, the appropriate matching conditions are

$$\sigma_{12}=0, \quad \sigma_{22}=\sigma_{22}^{(f)}, u_2=u_2^{(f)}, \quad T'=0 \quad at\ x_2=d \ , \tag{7}$$

where $T'\ \left(=T_{,x_2}\right)$ is the temperature gradient.

Analysis

We take the solution in the form

$$\left(u_1,u_2,T\right)=\left(U_1,U_2,\Theta\right)\exp[i\xi(x_1+\alpha x_2-ct)] \tag{8}$$

where ξ is the wave number U_1, U_2 and Θ are the constant amplitudes related to displacements and temperature, c is the phase velocity $\left(=\omega/\xi\right)$, ω is the circular frequency, α is the ratio of the x_2 and x_1 directions wave numbers. This choice of solutions leads to the coupled equations

$$M_{mn}(\alpha)U_n=0, \ m,n=1,2,3 \tag{9}$$

where

$$\begin{aligned} &M_{11}=1+c_2\alpha^2-\zeta^2, \quad M_{12}=c_3\alpha, \quad M_{13}=1, \\ &M_{22}=c_2+c_1\alpha^2-\zeta^2, \quad M_{23}=\bar{\beta}\alpha\ , \quad M_{31}=\varepsilon_1\omega_1^*\zeta^2\tau, \\ &M_{31}=\varepsilon_1\omega_1^*\zeta^2\tau\bar{\beta}\alpha, \quad M_{33}=1+\bar{K}\alpha^2-\omega_1^*\zeta^2\tau \end{aligned} \tag{10}$$

$$c_2=\frac{c_{66}}{c_{11}}, \quad c_1=\frac{c_{22}}{c_{11}}, \quad c_3=\frac{c_{12}+c_{66}}{c_{11}}, \ \varepsilon_1=\frac{\beta_1^2 \mathrm{T}_0}{\rho C_e c_{11}},$$

$$\zeta^2=\frac{\rho \mathrm{c}^2}{c_{11}}, \quad \omega_1^*=\frac{\mathrm{C_e} c_{11}}{K_1}, \quad \tau=\tau_0+{}^{\mathrm{i}}\!/\!_{\omega} \ , \ \bar{\beta}=\frac{\beta_2}{\beta_1}, \bar{K}=\frac{\bar{K}_2}{\bar{K}_1}$$

The system of equations (10) has a non-trivial solution if the determinant of the coefficients U_1, U_2 and Θ vanishes and this leads to the equation

$$\Delta a^6+B_1\alpha^4+B_2\alpha^2+B_3=0 \tag{11}$$

where

$$\begin{aligned} B_1 &=[-(c_1\bar{\beta}^2+c_1)c_2\omega_1^*\tau\zeta^2-(P\zeta^2+J)\bar{K}+c_1c_2] \\ B_2 &=[((\zeta^2-1)\bar{\beta}^2+2\bar{\beta}c_3-c_1)\varepsilon_1+(P\zeta^2+J)]\omega_1^*\tau\zeta \\ &\quad -(\zeta^2-1)(\zeta^2-c_2)\bar{K}-(P\zeta^2+J) \\ B_3 &=[(\zeta^2-1-\varepsilon_1)\omega_1^*\tau\zeta+(1-\zeta^2)](c_2-\zeta^2) \end{aligned}$$

$$P=c_1+c_2, \quad J=c_3^2-c_2^2-c_1, \quad \Delta=c_1c_2\bar{K}$$

where B_1, B_2 and B_3 which relates equation (11) admits six solutions for α (having the property $\alpha_2=-\alpha_1, \ \alpha_4=-\alpha_3, \ \alpha_6=-\alpha_5$. For each α_k , $k=1,2,\ldots 6,$ we can use the relations (9) and express the ratios

$$\gamma_k=\frac{U_2}{U_1}=\frac{M_{11}(\alpha_k)M_{33}(\alpha_k)-M_{13}(\alpha_k)M_{31}(\alpha_k)}{M_{13}(\alpha_k)M_{32}(\alpha_k)-M_{12}(\alpha_k)M_{33}(\alpha_k)},$$

$$\delta_k=\frac{\Theta}{U_1}=\frac{M_{31}(\alpha_k)M_{12}(\alpha_k)-M_{32}(\alpha_k)M_{11}(\alpha_k)}{M_{13}(\alpha_k)M_{32}(\alpha_k)-M_{12}(\alpha_k)M_{33}(\alpha_k)}. \tag{12}$$

Combining equations (8) and (12) with stress-strain and temperature relations (3) and using superposition, one obtain the formal solutions

$$\begin{bmatrix} u_1 \\ u_2 \\ T \\ \bar{\sigma}_{22} \\ \bar{\sigma}_{12} \\ T' \end{bmatrix}=\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ \gamma_1 & \gamma_2 & \gamma_3 & \gamma_4 & \gamma_5 & \gamma_6 \\ \delta_1 & \delta_2 & \delta_3 & \delta_4 & \delta_5 & \delta_6 \\ D_{11} & D_{12} & D_{13} & D_{14} & D_{15} & D_{16} \\ D_{21} & D_{22} & D_{23} & D_{24} & D_{25} & D_{26} \\ D_{31} & D_{32} & D_{33} & D_{34} & D_{35} & D_{36} \end{bmatrix}\begin{bmatrix} U_{11}E_1 \\ U_{12}E_2 \\ U_{13}E_3 \\ U_{14}E_4 \\ U_{15}E_5 \\ U_{16}E_6 \end{bmatrix}, \tag{13}$$

where

$$E_k=\exp\left(i\xi\alpha_k x_2\right), D_{1k}=(c_3-c_2)+c_1\alpha_k\gamma_k+i\xi^{-1}\beta_2\delta_k,$$

$$D_{2k}=c_2(\alpha_k+\gamma_k), \ D_{3k}=\alpha_k\delta_k \ \bar{\sigma}_{mn}=\frac{\sigma_{mn}}{i\xi}, \quad m,n=1,2. \tag{14}$$

Equation (13) , for the layer can be used to relate the displacements, temperature, stresses and heat flux at $x_2=0$ to those at $x_2=d$. This can be done by specializing (13) to $x_2=0$ and to $x_2=d$, and eliminating the common amplitude column made up $U_{11}, U_{12}, U_{13}, U_{14}, U_{15}$ and U_{16} of resulting in

$$F_{x_2=d}=(a_{ij})_{6\times 6}F_{x_2=0} \tag{15}$$

where $F=[u_1,u_2,T,\bar{\sigma}_{22},\bar{\sigma}_{12},T']^{\text{Transpose}}$.

Now, in order to satisfy the continuity conditions (6) and (7) at the substrate-plate and the plate -fluid interface, respectively, we need to solve the field equations in the substrate and in the fluid. By inspection, such solutions can be deduced and specialized from the formal solution (8), first, due to the absence of shear deformation, specializing (13) to the fluid half space and insuring boundedness for large values of x_2 yields

$$\begin{bmatrix} u_1^{(f)} \\ u_2^{(f)} \\ T_1^{(f)} \\ \sigma_{22}^{(f)} \end{bmatrix}=\begin{bmatrix} 1 & 1 \\ a_f & -a_f \\ T_1 & T_1 \\ \rho_f c^2 & \rho_f c^2 \end{bmatrix}\begin{bmatrix} 0 \\ U^{(f)}\exp[i\xi a_f x_2-d] \end{bmatrix}, \tag{16}$$

where $\alpha_f^2=\frac{\rho_f c_f^2}{\lambda_f}-1$.

Now specializing (13) to the substrate yields

$$\begin{bmatrix} u_1^{(s)} \\ u_2^{(s)} \\ T_2^{(s)} \\ \bar{\sigma}_{22}^{(s)} \\ \bar{\sigma}_{12}^{(s)} \\ T_2'^{(s)} \end{bmatrix}=\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ \gamma_1 & \gamma_2 & \gamma_3 & \gamma_4 & \gamma_5 & \gamma_6 \\ \delta_1 & \delta_2 & \delta_3 & \delta_4 & \delta_5 & \delta_6 \\ D_{11} & D_{12} & D_{13} & D_{14} & D_{15} & D_{16} \\ D_{21} & D_{22} & D_{23} & D_{24} & D_{25} & D_{26} \\ D_{31} & D_{32} & D_{33} & D_{34} & D_{35} & D_{36} \end{bmatrix}\begin{bmatrix} U_{11}^{(s)}E_1 \\ 0 \\ U_{11}^{(s)}E_3 \\ 0 \\ U_{11}^{(s)}E_5 \\ 0 \end{bmatrix}, \tag{17}$$

where $E_j=\exp\left(i\xi\alpha_j x_2\right)$.

Notice that in equation (17) the reflected wave amplitudes $U_{11}^{(s)}, U_{13}^{(s)}$ and $U_{15}^{(s)}$ vanish, since solutions must be bounded for large values of x2 in the substrate half space. By specializing (16) and (17) to the fluid interface $x_2=d$ and plate-substrate interface $x_2=0$, respectively, and followed by invoking the continuity conditions (6) and (7), we have

$$\begin{bmatrix} \alpha_f & -\alpha_f \\ T_1 & T_1 \\ \rho_f c^2 & \rho_f c^2 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}=\begin{bmatrix} R_{21} & R_{23} & R_{25} \\ R_{31} & R_{33} & R_{35} \\ R_{41} & R_{43} & R_{45} \\ R_{51} & R_{53} & R_{55} \\ R_{61} & R_{63} & R_{65} \end{bmatrix}\begin{bmatrix} U_{11}^{(s)} \\ U_{13}^{(s)} \\ U_{15}^{(s)} \end{bmatrix} \tag{18}$$

where $R_{ij}=[a_{ij}][b_{ij}]_s$ with $[b_{ij}]_s$ as the 6×6 matrix in equation (17).

For non-trivial solutions, the matrix equation (18) can be solved to give the surface waves characteristic equation in generalized thermoelasticity

$$G_1+G_2Q=0, \tag{19}$$

where

$$Q=\frac{\rho_f c^2}{\alpha_f}, \ G_1=\begin{bmatrix} 2R_{41} & 2R_{43} & 2R_{45} \\ R_{31}+R_{51} & R_{33}+R_{53} & R_{35}+R_{55} \\ 2R_{61} & 2R_{63} & 2R_{65} \end{bmatrix},$$

$$G_2=\begin{bmatrix} 2R_{21} & 2R_{23} & 2R_{25} \\ R_{31}+R_{51} & R_{33}+R_{31} & R_{53}+R_{55} \\ 2R_{61} & 2R_{63} & 2R_{65} \end{bmatrix}.$$

Special Cases

Dry Plate

In the absence of the fluid, i.e. for $\rho_f = 0$, and equation (19) reduces to

$$G_1 = 0, \qquad (20)$$

which defines the characteristic equation for the Rayleigh surface wave on the dry plate bonded to a semi-infinite solid substrate in the context of generalized theory of thermoelasticity.

Classical Case

When the thermoelastic coupling constant $\varepsilon_1 = 0$, then equation (11) reduces to

$$(\Delta'\alpha^4 + P_1\alpha^2 + Q_1)(1 + \bar{k}\alpha^2 - \omega_1^*\tau\zeta^2) = 0 \qquad (21)$$

where

$$P_1 = (1-\zeta^2)c_1 + (c_2 - \zeta^2)c_2$$

$$Q_1 = (1-\zeta^2)(c_2 - \zeta^2)$$

$$\Delta' = c_1 c_2 .$$

In this case thermal wave decoupled from its counterpart in elasticity.
The equation

$$1 + \bar{k}\alpha^2 - \omega_1^*\tau\zeta^2 = 0, \qquad (22)$$

defines the velocity and attenuation constant for the thermal wave. Clearly this is influenced by thermal relaxation time τ_0.
whereas

$$\Delta'\alpha^4 + P_1\alpha^2 + Q_1 = 0, \qquad (23)$$

corresponds to its counterpart in elasticity, which is in agreement with Rose, Nayfeh and Pilarski [11] .

Coupled Thermoelasticity

This case corresponds to situation when thermal relaxation time $\tau_0 = 0$. In this case the results obtained for generalized thermoelasticity reduce to the coupled theory of thermoelasticity.

Cubic and Isotropic Materials Symmetries

Results for thermoelastic solids possessing cubic and isotropic materials are the special cases of the general solutions obtained, as by invoking the appropriate restrictions on the thermoelastic properties.

Half Space

On setting either the thickness of the layer equal to zero or by setting the properties of the layer equal to corresponding properties of the substrate, results for half space can be deduced from the obtained results.

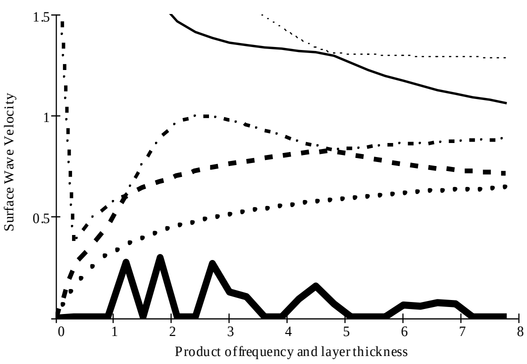

Figure 1. Dispersion curves for the first six modes of thermoelastic waves propagating modes in layered structure consisting of the upper layer from material I and the substrate material II with thermal relaxation time $\tau_0 = 4.0\times10^{-8}$

DISCUSSION

Dispersion curves for first five modes of surface waves propagating in a layered structure consisting of the upper layer from material I (magnesium) and the substrate from material II (Zinc) [11]. Figures illustrate the relationships between the surface wave velocity and the product of frequency and layer thickness. This figure gives some idea on how large the wavelength of the propagating surface wave should be with respect to the thickness of the layer in order to observe the differences in velocity related to the in-homogeneity generated by the layer thickness in generalized theory of thermoelasticity. It is also observed from the figure that as the wave length increases nearly 2.4 times the thickness of the plate d, the wave velocity of lower modes changes significantly. This demonstrate that how the wave velocity and its dependence on the wave length can be used to distinguish inhomogeneous character with the thickness. One can interchange the role of material I and the substrate from material II and can see the effect.

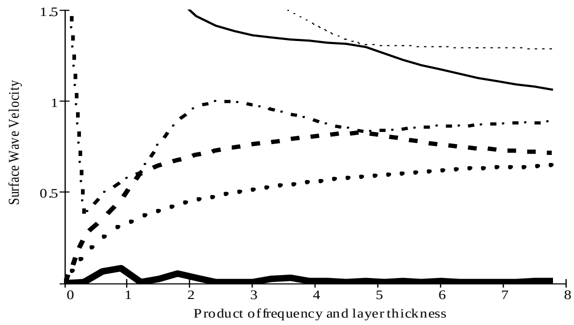

Figure 2. Dispersion curves for the first six modes of thermoelastic waves propagating modes in layered structure consisting of the upper layer from material I and the substrate material II with thermal relaxation time $\tau_0 = 4.0\times10^{-7}$

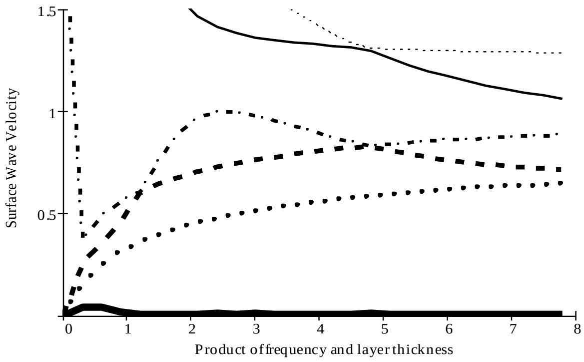

Figure 3. Dispersion curves for the first six modes of thermoelastic waves propagating modes in layered structure consisting of the upper layer from material I and the substrate material II with thermal relaxation time $\tau_0 = 4.0\times10^{-6}$

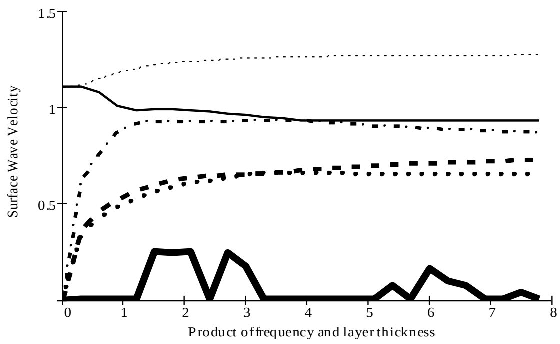

Figure 4. Propagating modes in layered structure consisting of the upper layer from material I and the substrate material II when thermal coupling constant equal to zero

At different values of thermal relaxation time from fijure 1 to 3, it is observed that as τ_0 increases, rapid change in the surface wave velocity is noticed in the lower modes whereas no significant change is observed in the higher modes. On considering taking thermal coupling constant equal to zero, results reduce to the classical theory figure 4. In figure 5, dispersion curves for the first lower six modes of thermoelastic waves propagating modes in layered structure consisting of the upper layer from material I and the substrate material II are shown, when thermal relaxation time is zero , in this case obtained results reduce for coupled thermoelasticity.

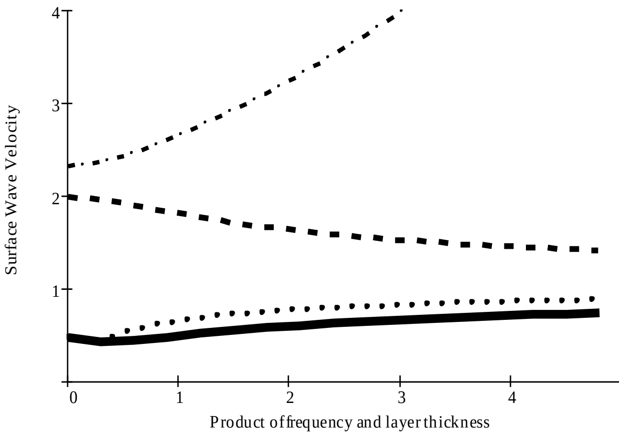

Figure 5. Dispersion curves for the first six modes of thermoelastic waves propagating modes in layered structure consisting of the upper layer from material I and the substrate material II when thermal relaxation time is zero

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